ALTERNATIVES TO THE DISCRETE FOURIER TRANSFORM

Doru Balcan, Aliaksei Sandryhaila, Jonathan Gross, Markus Püschel

Carnegie Mellon University
Pittsburgh, PA 15213

ABSTRACT

It is well-known that the discrete Fourier transform (DFT) of a finite length discrete-time signal samples the discrete-time Fourier transform (DTFT) of the same signal at equidistant points on the unit circle. Hence, as the signal length goes to infinity, the DFT approaches the DTFT. Associated with the DFT are circular convolution and a periodic signal extension. In this paper we identify a large class of alternatives to the DFT using the theory of polynomial algebras. Each of these transforms approaches the DTFT just as the DFT does, but has its own signal extension and own notion of convolution. Further, these transforms have Vandermonde structure, which enables their fast computation. We provide a few experimental examples that confirm our theoretical results.

***Index Terms*—** Discrete Fourier transforms, spectral analysis, boundary value problems, algebra, algebraic signal processing theory, Vandermonde matrix

1. INTRODUCTION

The discrete-time Fourier transform (DTFT) for a discrete-time signal with finite support $\mathbf{s} = (s_0, \dots, s_{n-1})$ is given by

$$y(\theta) = \sum_{0 \le \ell < n} s_\ell e^{-j\theta\ell}, \quad \theta \in [0, \pi). \tag{1}$$

Computing $y(\theta)$ is equivalent to evaluating the polynomial $s(x) = \sum_{0 \le \ell < n} s_\ell x^\ell$ on the unit circle $e^{-j\theta}$, $\theta \in [0, \pi)$.

A related, finite representation of **s** is computed via the discrete Fourier transform (DFT):

$$y_k = y(\tfrac{2\pi k}{n}) = \textstyle\sum_{0 \le \ell < n} s_\ell e^{-j\frac{2\pi k}{n}\ell}, \quad 0 \le k < n. \tag{2}$$

Computing $y(k)$ is now equivalent to evaluating $s(x)$ at the n nth roots of unity $e^{-2\pi kj/n}$, $0 \le k < n$, and shows that the DFT in (2) samples the DTFT in (1) at equidistant points on the unit circle. Hence, as n goes to infinity, the DFT approaches the DTFT. Further, it is well-known that applying the DFT assumes that the signal **s** is periodically extended and that the associated convolution becomes circular convolution.

Contribution. In this theoretical paper we derive a large set of alternatives to the DFT. Each of these transforms approaches the DTFT as n goes to infinity, has its own associated boundary condition and signal extension (which hence are not periodic), and own notion of convolution. Further, these transforms have Vandermonde structure, which enables their fast computation using $O(n \log^2(n))$ operations. For several examples, we experimentally confirm our theoretical result and show how they compare to the DFT when applied to a signal.

The derivation of the alternatives to the DFT makes use of the Beraha-Kahane-Weiss theorem [1] that describes the asymptotic behavior of root sets of polynomials. We combine this theorem with the theory of polynomial algebras [2], which is known to describe the DFT algebraically [3, 5]. This connection was recently extended in the algebraic signal processing theory [4].

Organization. Section 2 explains the polynomial algebra framework underlying both the DFT and the alternative transforms that we derive in this paper. This framework reduces the problem of deriving the alternative transforms to finding sequences of polynomials whose root sets converge to the unit circle. We identify a large class of such sequences in Section 3 and consider a few concrete examples for experiments in Section 4. We conclude with Section 5.

2. BACKGROUND

The key to deriving alternatives to the DFT is its interpretation in the framework of *polynomial algebras* $\mathbb{C}[x]/p_n(x)$, which we overview in this section. Every polynomial algebra has an associated notion of boundary condition, signal extension, convolution, spectrum, and Fourier transform, as explained in the algebraic signal processing theory [5, 4]. As running example, we use $\mathbb{C}[x]/(x^n - 1)$, which is known to be associated with the DFT [3].

In short, we will show in this paper that polynomials $p_n(x)$ other than $x^n - 1$ can be used to define alternatives to the DFT.

Polynomial algebra. An *algebra* is a vector space that is also a ring, i.e., permits the multiplication of its elements. Examples include the complex numbers $\mathbb{C}$ and the complex polynomials $\mathbb{C}[x]$.

Let $p_n(x) = x^n + \sum_{0 \le i < n} \beta_i x^i$ be a (normalized) polynomial of degree $\deg(p) = n$. The set of all polynomials of degree less than n,

$$\mathbb{C}[x]/p_n(x) = \{s(x) = \sum_{0 \le \ell < n} s_\ell x^\ell \mid \deg(s) < n\}$$

with addition and multiplication modulo $p(x)$ is called a *polynomial algebra*. As a vector space, $\mathbb{C}[x]/p(x)$ has dimension n. As a basis, we choose $b = (1, x, \dots, x^{n-1})$. For $s(x) \in \mathbb{C}[x]/p(x)$, we denote the list of coefficients with $\mathbf{s} = (s_0, \dots, s_{n-1})$.

$\mathbb{C}[x]/(x^n - 1)$ is an example of a polynomial algebra.

Boundary condition and signal extension. Every $\mathbb{C}[x]/p_n(x)$ has an associated (right) boundary condition which is obtained by reducing $x^n \bmod p_n(x) = -\sum_{0 \le i < n} \beta_i x^i$. Similarly, the (right) signal extension is given by reducing $x^m \bmod p_n(x)$ for $m \ge n$.

In our example, $p_n(x) = x^n - 1$, i.e., $x^n \bmod x^n - 1 = 1$ is the cyclic boundary condition. Further, $x^m \bmod x^n - 1 = x^{m \bmod n}$, i.e., a periodic signal extension.

Convolution. The convolution associated with $\mathbb{C}[x]/p_n(x)$ is the multiplication $h(x)s(x) \bmod p(x)$.

In our example $h(x)s(x) \bmod x^n - 1$ is equivalent to the circular convolution of the coordinate sequences **h** and **s** [3].

Spectrum and Fourier transform. We assume $p_n(x)$ has pairwise distinct zeros $\alpha = (\alpha_0, \dots, \alpha_{n-1})$. Then the Fourier transform associated with $\mathbb{C}[x]/p_n(x)$ is given by the Chinese remainder

This work was supported by NSF through award 0634967.

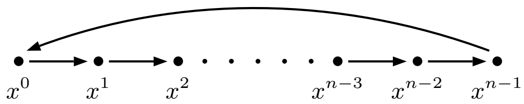


Fig. 1. The structure imposed on the signal by the polynomial algebra $\mathbb{C}[x]/(x^n-1)$ and hence by the DFT.

theorem [2], which decomposes it into a Cartesian product of one-dimensional polynomial algebras:

$$\begin{array}{rcl}\mathcal{F}:\ \mathbb{C}[x]/p_n(x) & \to & \bigoplus_{0\le k<n}\mathbb{C}[x]/(x-\alpha_k),\\ s(x) & \mapsto & (s(\alpha_0),\dots,s(\alpha_{n-1})).\end{array} \tag{3}$$

This $\mathcal{F}$ is a linear mapping (even an isomorphism of algebras) and $(s(\alpha_k))_{0\le k<n}$ is called the *spectrum* of $s(x)$. Hence, with respect to the basis b of $\mathbb{C}[x]/p_n(x)$ and $(x^0)=(1)$ in each of the $\mathbb{C}[x]/(x-\alpha_k)$ it is represented by a matrix (obtained by evaluating all basis elements in b at all zeros in α), which has Vandermonde structure:

$$\mathcal{F}=[\alpha_k^\ell]_{0\le k,\ell<n}. \tag{4}$$

Note that this class of transforms does not contain the discrete cosine and sine transforms, which can be captured in the algebraic framework by using Chebyshev polynomials [6, 4].

In our example, the zeros of x^n-1 are $\alpha_k=\omega_n^k$, $\omega_n=\exp(-2\pi j/n)$. Hence $\mathcal{F}=[\omega_n^{k\ell}]_{0\le k,\ell<n}=\mathrm{DFT}_n$ is exactly the discrete Fourier transform, i.e., the y_k in (2) are computed as $\mathbf{y}=\mathcal{F}\mathbf{s}$, $\mathbf{y}=(y_0,\dots,y_{n-1})$.

Visualization. The operation of x on the basis b of $\mathbb{C}[x]/p_n(x)$ can be represented by a graph. E.g., in our example $p_n(x)=x^n-1$, we obtain the directed circle in Fig. 1. Note how the graph captures the boundary condition $x^n=x^0$. Intuitively, the graph is the structure imposed on a signal $\mathbf{s}$ by the polynomial algebra.

Fast algorithms. Every general Fourier transform $\mathcal{F}$ in (4) is a Vandermonde matrix. Hence, $\mathbf{y}=\mathcal{F}\mathbf{s}$ can be computed using only $O(n\log^2(n))$ operations [7]. In the case of the DFT, even $O(n\log(n))$ is possible.

3. ALTERNATIVE DISCRETE FOURIER TRANSFORMS

Problem statement. We are interested in finding polynomial algebras $\mathbb{C}[x]/p_n(x)$ such that the set of zeros of p_n converges to the unit circle as n goes to infinity. The theory in Section 2 yields for each choice of $p_n(x)$ the associated notions of signal extension, convolution, spectrum, and Fourier transform. By construction, the latter will approach the DTFT in (1) as n goes to infinity, just as the DFT (which arises from the special case $p_n(x)=x^n-1$) in (2) does.

We will use the following definition.

Definition 1 Let $\{p_n(x)\mid n\ge 0\}$ be a sequence of complex polynomials of increasing degree $\deg(p_n)=n$. We say that $z\in\mathbb{C}$ is a *limit of zeros* for this sequence if there is a sequence $\{z_n\mid n\ge 0\}$ such that $p_n(z_n)=0$ and $\lim\limits_{n\to\infty}z_n=z$.

As an example, the limits of zeros of the sequence given by $p_n(x)=x^n-1$ are precisely all points on the unit circle. We note that we can extend the above definition to any sequence $\{q_n(x)\}$ of polynomials of increasing degrees *not necessarily equal* to their index.

Main theorem. The main result of this paper is the following theorem, which yields a large class of sequences of polynomials whose zero sets converge to the unit circle. We determine and experimentally test the associated alternatives to the DFT later.

Theorem 1 Let

$$q_n(x)=a_k(x)x^{kn}+a_{k-1}(x)x^{(k-1)n}+\dots+a_1(x)x^n+a_0(x), \tag{5}$$

where $a_i(x)\in\mathbb{C}[x]$ and $a_0,a_k\ne 0$. Then, $z\in\mathbb{C}$ is a limit of zeros if and only if one of the following holds:

(i) $|z|=1$.

(ii) $|z|<1$ and $a_0(z)=0$.

(iii) $|z|>1$ and $a_k(z)=0$.

In other words, Theorem 1 states that the limits of zeros of the polynomial sequence in (5) is the entire unit circle, plus possibly finitely many additional points, namely the roots of $a_0(x)$ inside the unit circle and the roots of $a_k(z)$ outside the unit circle.

This result can be readily extended by combining such families of polynomials, which yields the following corollary.

Corollary 1 Let $p_n(x)=\sum_{i=0}^k a_i(x)x^{\lfloor\frac{i(n-d)}{k}\rfloor}$ with $a_i(x)\in\mathbb{C}[x]$ and $a_0,a_k\ne 0$, $d=\deg(a_k)$. Then $z\in\mathbb{C}$ is a limit of zeros for this sequence if and only if one of *(i)–(iii)* in Theorem 5 holds.

$p_n(x)=x^n-1$ is a special case of the sequence in Corollary 1.

To prove Theorem 1, we use a theorem from Beraha, Kahane, and Weiss [1] explained next.

The Beraha-Kahane-Weiss theorem. Suppose $\{q_n\mid n\ge 0\}$ is a sequence of polynomials satisfying the m-th degree recursion

$$q_{n+m}(x)=-\sum_{j=1}^m f_j(x)q_{n+m-j}(x), \tag{6}$$

where the $f_j\in\mathbb{C}[x]$ are polynomials. For each $x\in\mathbb{C}$, (6) is an ordinary linear recurrence for the numbers $q_n(x)$, $n\ge 0$. With this observation, we can solve (6) following the standard procedure for linear recurrences [8], except that the results depend on x.

The *characteristic equation* associated with (6) is

$$Q_x(\lambda)=\lambda^m+\sum_{j=1}^m f_j(x)\lambda^{m-j}=0. \tag{7}$$

Let $\lambda_1(x),\dots,\lambda_m(x)$ be the m zeros of of Q_x. If the $\lambda_j(x)$ are pairwise distinct for a particular x, then $q_n(x)$ has the form

$$q_n(x)=\sum_{j=1}^m \alpha_j(x)\lambda_j(x)^n, \tag{8}$$

where the α_j are determined by solving a system of m linear equations obtained by letting $n=0,1,\dots,m-1$. If the $\lambda_j(x)$ are not pairwise distinct, (8) is adjusted in the usual way [8, appendix A].

We assume that the following two *nondegeneracy conditions* are satisfied:

- $\{q_n\}$ does not satisfy a recursion of degree less than m.
- There are no i,j such that $\lambda_i(x)\equiv\omega\lambda_j(x)$ for a constant ω with $|\omega|=1$.

Under these conditions, the following theorem holds.

Theorem 2 A point $z\in\mathbb{C}$ is a limit of zeros of $\{q_n\}$ if and only if the $\lambda_j(z)$ can be ordered such that one of the following holds:

(i) $|\lambda_1(z)|>|\lambda_j(z)|, 2\le j\le m$, and $\alpha_1(z)=0$.

(ii) $|\lambda_1(z)| = |\lambda_2(z)| = \ldots = |\lambda_l(z)| > |\lambda_j(z)|, l+1 \le j \le m$, for some $l \ge 2$.

Proof of Theorem 1. To apply Theorem 2, we show that our polynomial sequence (5)

- satisfies a linear recursion,
- allows for a simple computation of the roots $\lambda_j(x)$ and of the coefficients $\alpha_j(x)$ in (8), and
- satisfies the nondegeneracy conditions.

Lemma 1 Let $\{q_n(x)\}$ be the sequence defined in (5) and let $I = \{i \mid 0 \le i \le k, a_i(x) \ne 0\} = \{i_1, \ldots, i_m\}$, which we assume in increasing order. This implies $i_1 = 0$, $i_m = k$. Then $\{q_n(x)\}$ satisfies the following recurrence of order $m = |I|$, and no recurrence of smaller order:

$$q_n(x) = -\sum_{j=1}^{m} f_j(x) q_{n-j}(x), \qquad (9)$$

where the polynomials f_j are defined as

$$f_j(x) = (-1)^j \sum_{J \subset I, |J|=j} \prod_{\ell \in J} x^\ell. \qquad (10)$$

Further, the characteristic equation takes the simple form

$$Q_x(\lambda) = \lambda^m + \sum_{j=1}^{m} f_j(x) \lambda^{m-j} = \prod_{i \in I} (\lambda - x^i), \qquad (11)$$

which implies $\lambda_j(x) = x^{i_j}$; hence the nondegeneracy condition is satisfied. Comparing (8) with (5), this also shows $\alpha_j(x) = a_{i_j}(x)$.

In particular, the recurrence for the q_n does not depend on the $a_i(x)$ in (5); the a_i will affect only the initial conditions.

Proof. First we prove that $\{q_n(x)\}$ indeed satisfies the relation above by induction on $m = |I|$.

If $m = 1$ (implying $k = 0$) the statement holds, since $q_n(x) = a_0(x) = q_{n-1}(x)$, $f_1(x) = -1$, and $Q_x(\lambda) = \lambda - 1$.

Suppose now that the statement is true for $m - 1 \ge 1$. We will prove it also holds for m (implicitly, $m \ge 2$ and therefore $k > 0$). Let $I_k = I \setminus \{k\} \ne \emptyset$ (i.e., I without its largest element) and

$$r_{n-1}(x) = q_n(x) - x^k q_{n-1}(x) = \sum_{i \in I_k} b_i(x) x^{i(n-1)}, \qquad (12)$$

where for all $i \in I_k$, $b_i(x) = a_i(x)(x^i - x^k) \ne 0$. Define $\widetilde{f}_j(x) = (-1)^j \sum_{J \subset I_k, |J|=j} \prod_{\ell \in J} x^\ell$, where $1 \le j \le m-1$. As $|I_k| = m-1$, by applying the induction hypothesis to the sequence $\{r_n\}$, we find

$$\begin{aligned} r_{n-1}(x) &= -\sum_{j=1}^{m-1} \widetilde{f}_j(x) r_{n-1-j}(x) \\ &= -\widetilde{f}_1(x) q_{n-1}(x) - \sum_{j=2}^{m-1} (\widetilde{f}_j(x) - \widetilde{f}_{j-1}(x) x^k) q_{n-j}(x) \\ &\quad + \widetilde{f}_{m-1}(x) x^k q_{n-m}(x). \end{aligned}$$

We conclude the proof of the first claim in our lemma by observing that $-\widetilde{f}_1(x) = -f_1(x) - x^k$, $\widetilde{f}_{m-1}(x) x^k = -f_m(x)$, and $-\widetilde{f}_j(x) + \widetilde{f}_{j-1}(x) x^k = -f_j(x)$ for $1 < j < m$, which implies

$$q_n(x) = r_{n-1}(x) + x^k q_{n-1}(x) = -\sum_{j=1}^{m} f_j(x) q_{n-j}(x). \qquad (13)$$

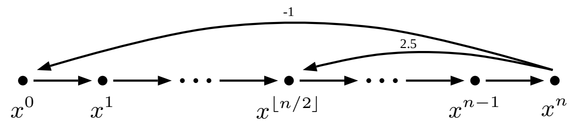


Fig. 2. The structure imposed on the signal by $\mathbb{C}[x]/(x^n - \frac{5}{2} x^{\lfloor n/2 \rfloor} + 1)$ and its associated Fourier transform.

To show that $\{q_n\}$ does not satisfy any recursion of order smaller than m, we use proof by contradiction but omit the details due to space limitations. $\square$

At this point we have shown that Theorem 2 is applicable to (5). To complete the proof of Theorem 1 we inspect which points satisfy one of the two conditions in Theorem 2. If for a $z \in \mathbb{C}$, exactly one of $|\lambda_j(z)| = |z^{i_j}|$ is maximal, then $|z| \ne 1$. In the case $|z| > 1$, we know $|z^k| > |z^i|$, for $i \in I \setminus \{k\}$, and so z is a limit of zeros for $\{q_n\}$ if and only if $a_k(z) = 0$. In the case $|z| < 1$, we have $1 = |z^0| > |z^i|$, for $i \in I \setminus \{0\}$ and z is a limit of zeros if and only if $a_0(z) = 0$. This completely handles the first condition in Theorem 2. Alternatively, if for $z \in \mathbb{C}$, there are $i, j \in I$, $i \ne j$, such that $|z^i| = |z^j|$, then necessarily $|z| = 1$. Since for all z on the unit circle $1 = |z^i|$, $i \in I$, we conclude that any such point is a limit of zeros for $\{q_n\}$. This completes the proof of Theorem 1.

Associated Fourier transforms. For each polynomial sequence p_n of the form considered above, and hence polynomial algebra $\mathbb{C}[x]/p_n$, the general theory from Section 2 provides the associated notions of boundary condition and signal extension (which will not be periodic in general), convolution, spectrum, and Fourier transform. The latter will be an alternative to the DFT, and has a fast algorithm due to its Vandermonde structure (Section 2).

4. EXAMPLE AND EXPERIMENTS

Example. As a first example, we consider the polynomials $p_n(x) = x^n - \frac{5}{2} x^{\lfloor n/2 \rfloor} + 1$, which match Corollary 1, and apply the theory in Section 2.

The boundary condition in $\mathbb{C}[x]/p_n$ is given by $x^n = \frac{5}{2} x^{\lfloor n/2 \rfloor} - 1$, which yields the visualization in Fig. 2. Convolution is the multiplication of polynomials $h(x)s(x) \bmod p_n(x)$. The Fourier transform $\mathcal{F}$ in (4) is determined by the zeros of p_n. For even $n = 2m$ they can be explicitly computed as

$$\alpha = (\sqrt[m]{2} w_m^{-m/2+k}, \ \frac{1}{\sqrt[m]{2}} w_m^{-m/2+k})_{1 \le k \le m},$$

where we ordered the zeros by increasing angle in $(-\pi, \pi]$. The root distribution for $n = 20$, 50, and 80 is shown in Fig. 3(b) below.

Hence the Fourier transform in the case $n = 2m$ becomes

$$\mathcal{F}_{2m} = [2^{\frac{(-1)^k l}{m}} w_m^{(\lfloor \frac{k}{2} \rfloor + 1 - \frac{m}{2}) l}]_{0 \le k, l < 2m}. \qquad (14)$$

Experiments. For our experiments, we consider four sequences of polynomials; the first is associated with the DFT:

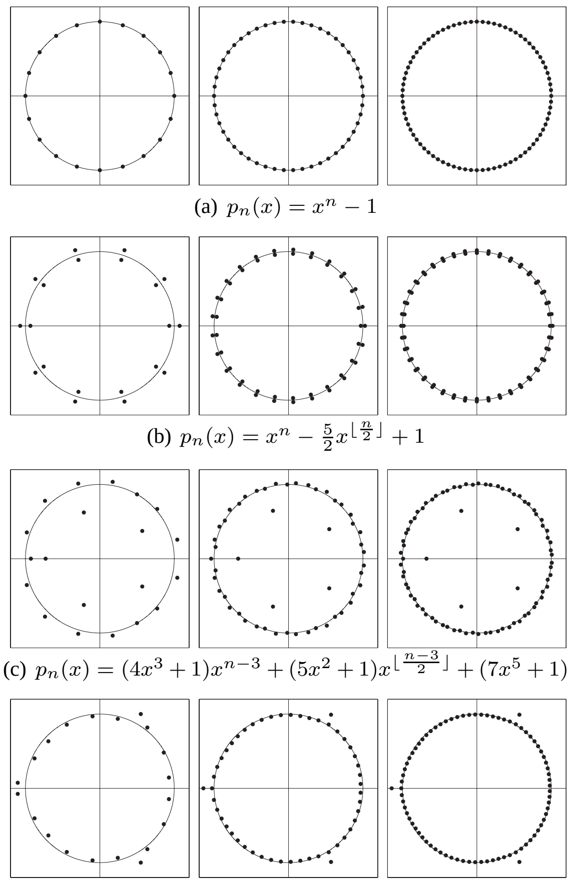
$$p_n(x) = x^n - 1, \qquad (15)$$

$$p_n(x) = x^n - \tfrac{5}{2} x^{\lfloor n/2 \rfloor} + 1, \qquad (16)$$

$$p_n(x) = (4x^3 + 1) x^{n-3} + (5x^2 + 1) x^{\lfloor \frac{n-3}{2} \rfloor} + (7x^5 + 1), \qquad (17)$$

$$p_n(x) = (2x^3 + 3) x^{n-3} - (x^5 - 2). \qquad (18)$$

In each case, we numerically compute the root set of $p_n(x)$ for $n \in \{20, 50, 80\}$, construct the corresponding Fourier transform,



(a) $p_n(x) = x^n - 1$

(b) $p_n(x) = x^n - \frac{5}{2}x^{\lfloor \frac{n}{2} \rfloor} + 1$

(c) $p_n(x) = (4x^3+1)x^{n-3} + (5x^2+1)x^{\lfloor \frac{n-3}{2} \rfloor} + (7x^5+1)$

(d) $p_n(x) = (2x^3+3)x^{n-3} - (x^5-2)$

Fig. 3. Roots of polynomials $p_n(x)$ for $n = 20, 50$, and 80.

and apply it to the first n coefficients of the sample signal shown in Fig. 4, which is one row of a gray-scale image.

According to Theorem 1, all roots of the polynomial sequences (15) and (16) converge to the unit circle. The sequence (17) has five limits of zeros inside the unit circle: $z_k = \sqrt[5]{1/7}e^{\pi i(2k+1)/5}$, $0 \leq k \leq 4$. The sequence (18) has three limits of zeros outside the unit circle: $z_k = \sqrt[3]{3/2}e^{\pi i(2k+1)/3}$, $0 \leq k \leq 2$. This is confirmed by Fig. 3, which shows the root sets for $n \in \{20, 50, 80\}$.

In each case we order the zeros of $p_n(x)$ by increasing angle in $(-\pi, \pi]$. For the DFT ($p_n(x) = x^n - 1$), this means that the y_k in (2) are ordered as $y_{\frac{n}{2}+1}, \dots y_{n-1}, y_0, \dots y_{\frac{n}{2}}$, i.e., the DC component is in the center.

Fig. 5(a)-5(b) shows, for $n \in \{20, 80\}$, the four Fourier transforms applied to the signal in Fig. 4. We observe that the spectra become similar for $n = 80$ as expected. In the last case, the three limits of zeros outside the unit circle make the three associated spectral values unbounded as n increases. In contrast, the five limits of zeros inside the unit circle in the third case do not cause this behavior.

5. CONCLUSION

The question we addressed in this paper is arguably fundamental to signal processing: why do we use a periodic signal extension and hence a DFT for finite length discrete-time signals? We showed that

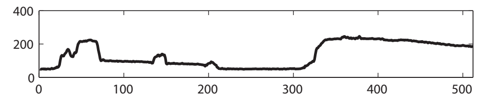
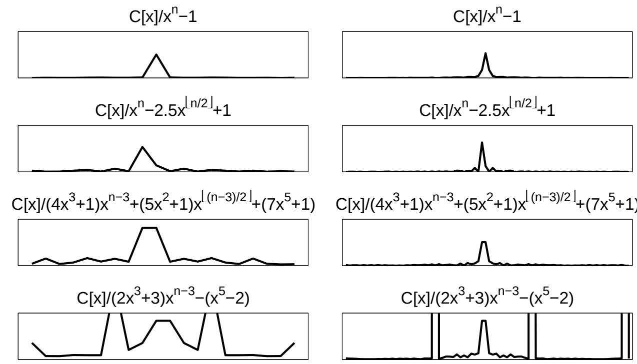


Fig. 4. Sample signal **s**.



(a) $n = 20$ (b) $n = 80$

Fig. 5. Magnitudes of the Fourier transform $\mathbf{y} = \mathcal{F}\mathbf{s}$ for $\mathbb{C}[x]/p_n(x)$, $n = 20, 80$, and **s** in Fig. 4.

if only asymptotic convergence to the DTFT is required, there are indeed many choices, each of which with its own signal extension and notion of convolution. Further, each of these alternative transforms possesses fast algorithms, which makes them in principle useful for applications. The question of these applications still remains.

6. REFERENCES

[1] S. Beraha, J. Kahane, and N. J. Weiss, "Limits of zeros of recursively defined families of polynomials," in *Studies in Foundations and Combinatorics: Advances in Math., Supplementary Studies*, vol. I, pp. 213–232. Academic Press, 1978.

[2] Paul A. Fuhrman, *A Polynomial Approach to Linear Algebra*, Springer Verlag, 1996.

[3] H. J. Nussbaumer, *Fast Fourier Transformation and Convolution Algorithms*, Springer, 2nd edition, 1982.

[4] M. Püschel and J. M. F. Moura, "Algebraic signal processing theory," available at http://arxiv.org/abs/cs.IT/0612077, parts of this manuscript are submitted as [5].

[5] M. Püschel and J. M. F. Moura, "Algebraic signal processing theory: Foundation and 1-D time," submitted for publication, part of [4].

[6] M. Püschel and J. M. F. Moura, "The algebraic approach to the discrete cosine and sine transforms and their fast algorithms," *SIAM Journal of Computing*, vol. 32, no. 5, pp. 1280–1316, 2003.

[7] V. Olshevsky and A. Shokrollahi, "Fast matrix-vector multiplication algorithms for confluent Cauchy-like matrices with applications," in *Proc. of the 32nd ACM Symposium on Theory of Computing (STOC'00)*, New York, 2000, pp. 573–581, ACM.

[8] S. Lipschutz and M. Lipson, *Discrete Mathematics*, McGraw Hill, 2 edition, 1997.