THE DISCRETE TRIANGLE TRANSFORM

Markus Püschel

Dept. of Electrical and Computer Engineering
Carnegie Mellon University, Pittsburgh, U.S.A.

Martin Rötteler

Dept. of Combinatorics and Optimization
University of Waterloo, Waterloo, Canada

ABSTRACT

We introduce the discrete triangle transform (DTT), a non-separable transform for signal processing on a two-dimensional equispaced triangular grid. The DTT is, in a strict mathematical sense, a generalization of the DCT, type III, to two dimensions, since the DTT is built from Chebyshev polynomials in two variables in the same way as the DCT, type III, is built from Chebyshev polynomials in one variable. We provide boundary conditions, signal extension, and diagonalization properties for the DTT. Finally, we give evidence that the DTT has Cooley-Tukey FFT like algorithms that enable its efficient computation.

1. INTRODUCTION

Recent research has shown that the 16 types of discrete cosine and sine transforms (DCTs and DSTs), can, like the discrete Fourier transform (DFT), be characterized in the framework of polynomial algebras in one variable [1, 2]. The polynomial algebra for a DCT or DST explains the interaction between the underlying signal model, its boundary conditions (b.c.), its signal extension, and gives easy access to the transform's properties, e.g., a characterization of the matrices diagonalized by it. Further, and perhaps most importantly, it provides the means to concisely derive the transform's known and even new [3] fast algorithms by manipulating the polynomial algebra rather than the matrix entries.

The polynomial algebras associated with the DCTs and DSTs are built in different ways from Chebyshev polynomials in one variable. The underlying signal model in each case can be visualized by an undirected line-shaped graph with loops at each side representing the specific b.c. For example, for the DCT, type III, this graph is given in Fig. 1 and will be explained later. (For the DFT, the analogous graph is the well-known directed graph arising from an n-gon, which captures the cyclic b.c.)

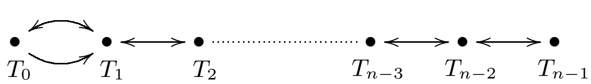


Fig. 1. Signal model associated with the DCT, type III, with algebra $\mathbb{C}[x]/T_n(x)$ and basis $b = (T_0, \dots, T_{n-1})$.

For two-dimensional signals (e.g., images) Kronecker (tensor) products of one-dimensional transforms are usually used, for example $\mathrm{DCT}^{(\mathrm{III})} \otimes \mathrm{DCT}^{(\mathrm{III})}$, i.e., these transforms are separable. The underlying polynomial algebra is accordingly just the tensor product of the one-dimensional algebras and the associated graph is the direct product of the one-dimensional graphs, e.g., a torus for the 2-d DFT or, for the 2-d $\mathrm{DCT}^{(\mathrm{III})}$, the graph in Fig. 2 (shown for size 8×8), which is a square grid; each inner point has four neighbors.

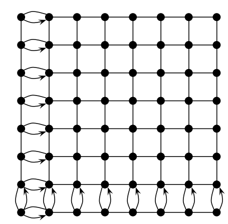


Fig. 2. Signal model associated with the 2-d DCT, type III, with algebra $\mathbb{C}[x, y]/\langle T_n(x), T_n(y)\rangle = \mathbb{C}[x]/T_n(x) \otimes \mathbb{C}[y]/T_n(y)$.

In this paper, we present a different method that yields a *non-separable* two-dimensional transform using the Chebyshev polynomials in *two variables*, which are far less well-known than their one-dimensional counterparts. Analogous to the $\mathrm{DCT}^{(\mathrm{III})}$'s one-dimensional polynomial algebra, we construct a two-dimensional polynomial algebra, which, as it turns out, provides the structure of a triangular grid with simple b.c., and in which each inner point has six neighbors, see Fig. 3 (the details are explained later). Thus, we call the associated transform *discrete triangular transform* (DTT).

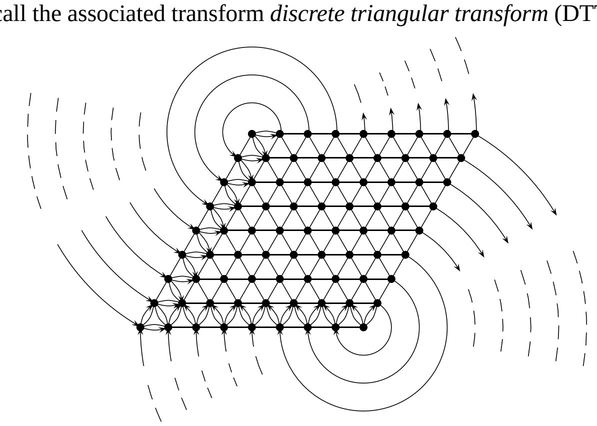


Fig. 3. Signal model associated to the DTT with algebra $\mathbb{C}[x, y]/\langle C_n(x, y), \overline{C}_n(x, y)\rangle$.

Organization. In Section 2 we give the necessary background of polynomial algebras and transforms in one variable and develop the special case of the DCT, type III, in Section 3. Section 4 briefly discusses two-dimensional polynomial algebras, including the non-separable case. In Section 5 we derive the DTT in analogy to Section 3. We discuss the DTT's properties and show how to derive a Cooley-Tukey FFT like algorithm.

The work of Markus Püschel was supported by NSF award 0310941; the work of Martin Rötteler was supported by CFI, ORDFC, and MITACS.

2. POLYNOMIAL ALGEBRAS AND TRANSFORMS: ONE VARIABLE

In this section we provide the mathematical background on polynomial algebras in one variable and their associated polynomial transforms. This theory provides the foundation for understanding the DFT, the DCTs and DSTs, their properties, their associated signal models, and their fast algorithms [1, 2]. The particular case DCT, type III, will be explained in Section 3.

Algebra. A vector space $\mathcal{A}$ that also permits multiplication of elements such that the distributive law holds is called an *algebra*. Examples include the set of complex numbers $\mathbb{C}$ and the sets $\mathbb{C}[x]$ or $\mathbb{C}[x, y]$ of complex polynomials in one or two variables, respectively.

Polynomial algebra. Let $p(x)$ be a polynomial of degree $\deg(p) = n$. Then, $\mathcal{A} = \mathbb{C}[x]/p(x) = \{q(x) \mid \deg(q) < n\}$, the set of residue classes modulo p, is an n-dimensional algebra with respect to the addition of polynomials and the multiplication of polynomials modulo p. We call $\mathcal{A}$ a *polynomial algebra*.

Polynomial transform. We assume that $\mathcal{A} = \mathbb{C}[x]/p(x)$ and that $p(x)$ has pairwise distinct zeros $\alpha = (\alpha_0, \dots, \alpha_{n-1})$. Then, the Chinese Remainder Theorem (CRT) decomposes $\mathcal{A}$ into a Cartesian product of one-dimensional polynomial algebras as

$$\begin{aligned} \mathbb{C}[x]/p(x) &\rightarrow \mathbb{C}[x]/(x-\alpha_0) \oplus \dots \oplus \mathbb{C}[x]/(x-\alpha_{n-1}), \\ q(x) &\mapsto (q(\alpha_0), \dots, q(\alpha_{n-1})). \end{aligned} \qquad (1)$$

The right hand side of (1) is called the spectrum of $\mathcal{A} = \mathbb{C}[x]/p(x)$ and the spectrum of $q \in \mathcal{A}$, respectively. We choose an arbitrary basis $b = (p_0, \dots, p_{n-1})$, $\deg(p_i) < n$, for $\mathcal{A}$, and $b_k = (x^0)$ as the basis for $\mathbb{C}[x]/(x - \alpha_k)$, $0 \le k < n$. Then the decomposition in (1) is given by the *polynomial transform*

$$\mathcal{P}_{b,\alpha} = [p_\ell(\alpha_k)]_{0 \le k, \ell < n}. \qquad (2)$$

which has as entries the projections of $p_\ell \in b$ onto the spectrum $\mathbb{C}[x]/(x - \alpha_k)$, $0 \le k < n$: $p_\ell(x) \equiv p_\ell(\alpha_k) \bmod (x - \alpha_k)$.

Shift and diagonalization property. Let $q(x), r(x) \in \mathcal{A} = \mathbb{C}[x]/p(x)$. Because of $x(aq(x)) = axq(x)$, $a \in \mathbb{C}$, and the distributive law $x(q(x) + r(x)) = xq(x) + xr(x)$, the multiplication by x is a linear mapping in $\mathcal{A}$, and thus, w.r.t. a chosen basis b, is represented by a matrix M_x. We call x the *shift* in $\mathcal{A}$ (with basis b) and M_x its matrix version. We will visualize the signal model associated to a polynomial algebra by the graph defined by M_x. Further, M_x is diagonalized by $\mathcal{P}_{b,\alpha}$, namely

$$\mathcal{P}_{b,\alpha} \cdot M_x \cdot \mathcal{P}_{b,\alpha}^{-1} = \operatorname{diag}(\alpha_0, \dots, \alpha_{n-1}).$$

Fast Algorithms. There are various ways of deriving fast algorithms for a polynomial transform [1]. In each case, the algorithms are derived by decomposing the underlying algebra in steps, rather than manipulating the transform's matrix entries. In [3] we have shown that Cooley-Tukey FFT type algorithms are obtained in the special case where $p(x) = q(r(x))$ *decomposes* into two polynomials q and r. This was used to provide a simple derivation of the Cooley-Tukey FFT and its counterparts for the DCTs of type III and II, which have not been known before.

3. EXAMPLE: DCT, TYPE III

We characterize the DCT, type III, in the above framework.

Chebyshev polynomials (one variable). First we need to introduce the Chebyshev polynomials in one variable [4], which are recursively defined by

$$T_0 = 1,\ T_1 = x, \quad T_n = 2xT_{n-1} - T_{n-2}, \quad n > 1. \qquad (3)$$

The recursion can be reversed to compute T_n for $n < 0$. A parameterization of T_n is given by

$$T_n = (u^n + u^{-n})/2, \quad x = (u + u^{-1})/2. \qquad (4)$$

By setting $u = e^{j\theta}$, we obtain the trigonometric form

$$T_n = \cos n\theta, \quad x = \cos\theta,\ n \in \mathbb{Z}, \qquad (5)$$

The following properties of the Chebyshev polynomials can be derived from (5):

$$T_{-n} = T_n, \qquad (6)$$

$$T_k T_n = (T_{n+k} + T_{n-k})/2, \qquad (7)$$

$$T_{km} = T_k(T_m), \qquad (8)$$

$$n \text{ zeros of } T_n:\ \alpha_k = \cos(k + 1/2)\pi/n,\ 0 \le k < n. \qquad (9)$$

DCT, type III. The DCT, type III, is a polynomial transform for $\mathcal{A} = \mathbb{C}[x]/T_n(x)$ with basis $b = (T_0, \dots, T_{n-1})$. Namely, since the zeros of T_n are given by (9), and using (5), we get

$$\mathrm{DCT}_n^{\text{(III)}} = \mathcal{P}_{b,\alpha} = [T_\ell(\cos(k{+}1/2)\pi/n)] = [\cos \ell(k{+}1/2)\pi/n]$$

as desired. Knowing that $\mathcal{A} = \mathbb{C}[x]/T_n(x)$ is the underlying polynomial algebra for the $\mathrm{DCT}^{\text{(III)}}$, we can easily read off properties of the associated signal model and the transform itself. We briefly summarize the most important properties, which will have precise counterparts for the discrete triangle transform introduced later.

Boundary conditions (b.c.). The basis of $\mathcal{A}$ are the polynomials T_ℓ, $0 \le \ell < n$. Thus, the left boundary is T_{-1}. Using (6), we get the left b.c. $T_{-1} = T_1$. Similarly, the right boundary is T_n, which in $\mathcal{A}$ is equal to 0, i. e., the right b.c. is $T_n = 0$.

Signal extension. The left signal extension associated to $\mathcal{A}$ is given by (6) as

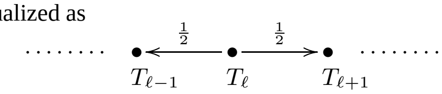
$$T_{-\ell} = T_\ell, \quad \ell > 0, \qquad (10)$$

i.e., symmetric with symmetry point T_0. The right signal extension is obtained by multiplying the right b.c. with T_ℓ using (7) to get

$$T_{n+\ell} = -T_{n-\ell}, \quad \ell > 0, \qquad (11)$$

i.e., it is antisymmetric with symmetry point $T_n = 0$.

Shift. The operation of the shift $T_1 = x$ on b is determined by the recursion in (3), namely $xT_\ell = (T_{\ell+1} + T_{\ell-1})/2$, which can be visualized as



(12)

With respect to the basis b the shift x is given by a matrix M_x. This matrix M_x is diagonalized by $\mathrm{DCT}_n^{\text{(III)}}$.

Visualization of the signal model. Combining the graph segments in (12) for all ℓ (dropping the $1/2$'s) and including the left and right boundary conditions, we obtain the graph in Fig. 1, which visualizes the signal model, i.e., the operation of the shift, associated with $\mathcal{A}$ and having M_x as adjacency matrix.

Fast algorithm. Based on property (8) we can derive fast algorithms for $\mathrm{DCT}^{\text{(III)}}$ [3]. We briefly sketch a special case that parallels the algorithm for the discrete triangle transform to be derived later.

Let $n = 2m$ and thus $T_n = T_2(T_m)$ by (8). The algorithm derivation follows the following stepwise decomposition, which first factorizes the outer polynomial $T_2(x) = (x - \cos(\pi/4))(x - \cos(3\pi/4))$, then the remaining polynomials of degree m, followed by a reordering of the spectrum.

$$\begin{aligned}
& \mathbb{C}[x]/T_2(T_m) \\
\to\ & \mathbb{C}[x]/(T_m - \cos\tfrac{\pi}{4}) \oplus \mathbb{C}[x]/(T_m - \cos\tfrac{3\pi}{4}) \\
\to\ & \bigoplus_{\substack{0\le k<m \\ k\equiv\pm1 \bmod 8}} \mathbb{C}[x]/(x - \cos\tfrac{k\pi}{2n}) \oplus \bigoplus_{\substack{0\le k<m \\ k\equiv\pm3 \bmod 8}} \mathbb{C}[x]/(x - \cos\tfrac{k\pi}{2n}) \\
\to\ & \bigoplus_{0\le k<n} \mathbb{C}[x]/(x - \cos\tfrac{2k+1}{2n}\pi)
\end{aligned}$$

The resulting algorithm has the form

$$\begin{aligned}
\mathrm{DCT}_n^{(\mathrm{III})} = P_n(&\mathrm{DCT}_m^{(\mathrm{III})}(\cos\tfrac{\pi}{4}) \\
&\oplus \mathrm{DCT}_m^{(\mathrm{III})}(\cos\tfrac{3\pi}{4}))(\mathrm{DCT}_2^{(\mathrm{III})} \otimes \mathrm{I}_m)B_n, \quad (13)
\end{aligned}$$

where B_n is an initial base change and the other three factors (from left to right) correspond to the three steps in the derivation. P_n is a permutation. The exact form can be found in [3]. The transforms $\mathrm{DCT}_m^{(\mathrm{III})}(\cos r\pi)$ are "skew" $\mathrm{DCT}^{(\mathrm{III})}$'s, i.e., polynomial transforms for the algebra $\mathbb{C}[x]/(T_m - \cos(r\pi))$ with basis $b = (T_0, \dots, T_{m-1})$. They can be further decomposed similarly to (13), since $T_m - \cos(r\pi)$ decomposes if T_m does. The resulting algorithm for the $\mathrm{DCT}_n^{(\mathrm{III})}$ has best-known arithmetic cost [3].

4. POLYNOMIAL ALGEBRAS AND TRANSFORMS: TWO VARIABLES

The notion of polynomial algebras and transforms (Section 2) can be extended to the case of two variables. Instead of operating modulo one polynomial we now require two polynomials, $p(x, y)$ and $q(x, y)$, in two variables, both of degree n. The polynomial algebra is now written as $\mathcal{A} = \mathbb{C}[x, y]/\langle p(x, y), q(x, y)\rangle$. In the generic case, $\mathcal{A}$ is a finite-dimensional vector space and there are n^2 distinct common zeros $\alpha = ((\mu_0, \nu_0), \dots, (\mu_{n^2-1}, \nu_{n^2-1})$ of p and q. Hence, in analogy to the one-variable case, $\mathcal{A}$ is decomposed by the Chinese Remainder Theorem as

$$\mathbb{C}[x, y]/\langle p(x, y), q(x, y)\rangle \to \bigoplus_{0\le i<n^2} \mathbb{C}[x, y]/\langle x - \mu_i, y - \nu_i\rangle$$

into a Cartesian product of one-dimensional algebras, the spectrum of $\mathcal{A}$.

If we choose a basis b of $\mathcal{A}$ and denote the common zeros by α then this decomposition is given by the polynomial transform $\mathcal{P}_{b,\alpha}$ defined as in (2).

Now, the multiplications by x and by y are both linear mappings in $\mathcal{A}$. Hence both can be represented by matrices with respect to the basis b. Note that the obtained matrices M_x and M_y are both diagonalized by the transform $\mathcal{P}_{b,\alpha}$. In $\mathcal{A}$ we now have *two* shifts, x and y, with associated matrices M_x and M_y, which are simultaneously diagonalized by $\mathcal{P}_{b,\alpha}$.

A simple way to construct a polynomial algebra in two variables is as a tensor product of two copies of a one-variable $\mathcal{A} = \mathbb{C}[x]/p(x)$ with basis b and polynomial transform $\mathcal{P}_{b,\alpha}$, namely

$$\mathbb{C}[x]/p(x) \otimes \mathbb{C}[y]/p(y) \cong \mathbb{C}[x, y]/\langle p(x), p(y)\rangle, \qquad (14)$$

which is called *separable*. The basis for this algebra is just the Cartesian product of b with itself, and the polynomial transform of (14) is the Kronecker product $\mathcal{P}_{b,\alpha} \otimes \mathcal{P}_{b,\alpha}$. This construction is used in signal processing to construct higher-dimensional transforms. Further, the visualizing graph is just the direct product of the graph for $\mathcal{A}$ with itself. For example, Figure 2 shows the graph for a two-dimensional $\mathrm{DCT}_8^{(\mathrm{III})}$, arising as a direct product of Fig. 1 with itself, i.e., every row and column in Fig. 2 is a copy of Fig. 1.

The two-dimensional transform introduced next is not separable.

5. DISCRETE TRIANGLE TRANSFORM

In this section we introduce the discrete triangle transform, which is built from Chebyshev polynomials in *two variables* in an analogous way as the DCT, type III, is built from Chebyshev polynomials in one variable. Thus, we chose the following presentation to parallel Section 3.

To avoid confusion, we denote the Chebyshev polynomials in two variables by the letter C. Further, if $p(x, y) \in \mathbb{C}[x, y]$, then we denote by $\overline{p}(x, y) = p(y, x)$ the polynomial with reversed arguments.

Chebyshev polynomials (two variables). We define the Chebyshev polynomials $C_n = C_n(x, y)$ in two variables by the recursion

$$\begin{gathered}
C_{-1} = y,\ C_0 = 1,\ C_1 = x, \\
C_n = 3xC_{n-1} - 3yC_{n-2} + C_{n-3}, \quad n > 1.
\end{gathered}$$

The recursion can be reversed to compute C_n for $n < 1$. Next, we extend this definition to obtain the full two-dimensional grid of Chebyshev polynomials $C_{n,m}$, for $n, m \in \mathbb{Z}$, by setting

$$\begin{gathered}
C_{n,0} = C_n, \quad C_{0,n} = \overline{C}_n, \\
C_{n,m} = (3C_n\overline{C}_m - C_{n-m})/2. \qquad (15)
\end{gathered}$$

This definition is a slight modification of the definition in [5].

The following parameterization of $C_{n,m}$ corresponding to (4) can be found in [5].

$$\begin{aligned}
C_{n,m}(x, y) = \tfrac{1}{6}(&u^n v^{-m} + u^{-m} v^n + u^{n+m} v^m \\
&+ u^m v^{n+m} + u^{-n-m} v^{-n} + u^{-n} v^{-n-m}), \quad (16)
\end{aligned}$$

$$x = \tfrac{1}{3}(u + v + (uv)^{-1}), \quad y = \tfrac{1}{3}(u^{-1} + v^{-1} + uv)/3. \quad (17)$$

In particular,

$$C_n = \tfrac{1}{3}(u^n + v^n + (uv)^{-n}), \quad \overline{C}_n = \tfrac{1}{3}(u^{-n} + v^{-n} + (uv)^n).$$

Substituting $u = e^{j\theta}$, $v = e^{j\eta}$ yields the analogue of (5).

The following four properties can be derived from (16) and are the analogue of (6)–(9); (20) is due to [6].

$$C_{n,-m} = C_{n-m,m}, \quad C_{-n,m} = C_{n,m-n} \qquad (18)$$

$$\begin{aligned}
C_k C_{n,m} &= \tfrac{1}{3}(C_{n-k,m+k} + C_{n,m-k} + C_{n+k,m}) \\
\overline{C}_k C_{n,m} &= \tfrac{1}{3}(C_{n-k,m} + C_{n+k,m-1} + C_{n,m+k})
\end{aligned} \qquad (19)$$

$$C_{km} = C_k(C_m, \overline{C}_m), \quad \overline{C}_{km} = \overline{C}_k(C_m, \overline{C}_m) \qquad (20)$$

$$\begin{gathered}
n^2 \text{ zeros of } C_n = \overline{C}_n = 0\ (0 \le k, \ell < n): \\
(u_k, v_\ell) = (\omega_n^k, \omega_{3n}^{1+3\ell}), \quad \omega_n = e^{-2\pi j/n}.
\end{gathered} \qquad (21)$$

The zeros of $C_n = \overline{C}_n = 0$ are pairwise distinct and given in terms of (u, v) in the parameterization (17).

Discrete triangle transform. We define the discrete triangle transform $\mathrm{DTT}_{n\times n}$ for input size $n \times n$ as the polynomial transform for the polynomial algebra $\mathbb{C}[x, y]/\langle C_n, \overline{C}_n\rangle$ with basis $b = (C_{k,\ell} \mid 0 \le k, \ell < n)$, and list of zeros $\alpha = ((\alpha_{i,j}, \beta_{i,j}) \mid$

$0 \leq i,j < n$), where $\alpha_{i,j}$ and $\beta_{i,j}$ are determined by $(u_i, v_j) = (\omega_n^i, \omega_{3n}^{1+3j})$ in the parameterization of x and y in (17). In other words, $\mathrm{DTT}_{n\times n}$ is the $n^2 \times n^2$ matrix given by

$$\mathrm{DTT}_{n\times n} = [C_{k,\ell}(\alpha_{i,j}, \beta_{i,j})]_{0\leq i,j<n,\ 0\leq k,\ell<n}.$$

The double index (i,j) is the row index, and (k,ℓ) is the column index, both ordered lexicographically. $C_{k,\ell}(\alpha_{i,j}, \beta_{i,j})$ is evaluated by substituting $(u_i, v_j) = (\omega_n^i, \omega_{3n}^{3j+1})$ in (16).

The smallest example is $n = 2$. We have $C_{0,0} = 1, C_{0,1} = y, C_{1,0} = x, C_{1,1} = \frac{1}{2}(3xy - 1)$; the zeros of $C_2 = \overline{C}_2 = 0$, i.e., $3x^2 - 2y = 3y^2 - 2x = 0$, are given by (in this order)

$$(\tfrac{2}{3}, \tfrac{2}{3}), (0,0), (\tfrac{2}{3}\omega_3, \tfrac{2}{3}\omega_3^2), (\tfrac{2}{3}\omega_3^2, \tfrac{2}{3}\omega_3).$$

Thus, $\mathrm{DTT}_{2\times 2}$ is the 4×4 matrix

$$\mathrm{DTT}_{2\times 2} = \begin{bmatrix} 1 & \frac{2}{3} & \frac{2}{3} & \frac{1}{6} \\ 1 & 0 & 0 & -\frac{1}{2} \\ 1 & \frac{2}{3}\omega_3^2 & \frac{2}{3}\omega_3 & \frac{1}{6} \\ 1 & \frac{2}{3}\omega_3 & \frac{2}{3}\omega_3^2 & \frac{1}{6} \end{bmatrix}.$$

Boundary conditions. The basis of $\mathcal{A}$ consists of the polynomials $C_{k,\ell}$, $0 \leq k, \ell < n$, which we assume to be arranged in a two-dimensional coordinate system. Then the bottom and left boundary are the polynomials $C_{-1,\ell}$ and $C_{k,-1}$. Using (18),

$$\text{bottom b.c.: } C_{k,-1} = C_{k-1,1}, \quad \text{left b.c.: } C_{-1,\ell} = C_{1,\ell-1}.$$

Similarly, the upper and right boundary are the polynomials $C_{k,n}$ and $C_{n,\ell}$, respectively. Since in $\mathcal{A}$ the equations $C_n = \overline{C}_n = 0$ hold, and using (15) and (18), we obtain

$$\text{upper b.c.: } C_{k,n} = -\tfrac{1}{2}C_{0,n-k}, \quad \text{right b.c.: } C_{n,\ell} = -\tfrac{1}{2}C_{n-k,0}. \tag{22}$$

Signal extension. The bottom and left signal extensions, respectively, are given by (18) as

$$\text{bottom: } \quad C_{k,-\ell} = C_{k-\ell,\ell}, \quad \text{left: } \quad C_{-k,\ell} = C_{k,k-\ell},$$

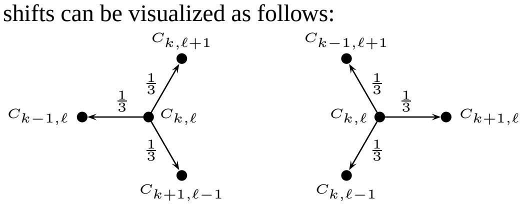
which turns out to be symmetric w.r.t. the coordinate axes (see the visualization below, which requires an angle of 60 degrees between the coordinate axes). The upper and right signal extensions are obtained by multiplying (22) with C_k and $\overline{C}_k$ and using (19):

$$\begin{aligned} &\text{upper:} \quad C_{k,n+\ell} = -C_{k+\ell,n-\ell} - C_{\ell,n-k-\ell}, \\ &\text{right:} \quad C_{n+k,\ell} = -C_{n-k,\ell+k} - C_{n-\ell-k,k}. \end{aligned}$$

Shift. We have two shifts, x and y. Their operation is a special case of (19) for $k = 1$ and given by

$$\begin{aligned} xC_{k,\ell} &= \tfrac{1}{3}(C_{k-1,\ell+1} + C_{k,\ell-1} + C_{k+1,\ell}) \\ yC_{k,\ell} &= \tfrac{1}{3}(C_{k-1,\ell} + C_{k+1,\ell-1} + C_{k,\ell+1}) \end{aligned}$$

The shifts can be visualized as follows:



With respect to the basis b, the shifts x and y are given by matrices M_x and M_y, respectively. The DTT simultaneously diagonalizes these matrices.

Visualization of signal model. Combining the above graph segments (dropping the 1/3's) and including the boundary conditions yields the graph in Fig. 3 (shown for size 8×8), which visualizes the signal model associated to $\mathcal{A}$. Note that the grid naturally becomes triangular to preserve the equidistance of the shifts. The graph encodes the left and bottom b.c. analogous to Fig. 2 by additional small arrows emanating from these boundaries. The upper and right b.c. connect to the left and bottom boundary, respectively.

Fast algorithm. Based on the decomposition property (20) of Chebyshev polynomials in two variables, we design a fast algorithm for the $\mathrm{DTT}_{n\times n}$. Due to space limitations we only state the decomposition of the algebra and the corresponding factorization of the transform.

$$\begin{aligned}
& \mathbb{C}[x,y]/\langle C_2(C_m, \overline{C}_m), \overline{C}_2(C_m, \overline{C}_m)\rangle \\
\to \quad & \mathbb{C}[x,y]/\langle C_m - \tfrac{2}{3}, \overline{C}_m - \tfrac{2}{3}\rangle \\
& \oplus \mathbb{C}[x,y]/\langle C_m, \overline{C}_m\rangle \\
& \oplus \mathbb{C}[x,y]/\langle C_m - \tfrac{2}{3}\omega_3, \overline{C}_m - \tfrac{2}{3}\omega_3^2\rangle \\
& \oplus \mathbb{C}[x,y]/\langle C_m - \tfrac{2}{3}\omega_3^2, \overline{C}_m - \tfrac{2}{3}\omega_3\rangle \\
\to \quad & \bigoplus_{0\leq i,j<m} \mathbb{C}[x,y]/\langle x - \alpha_{2i,2j}, y - \beta_{2i,2j}\rangle \\
& \oplus \bigoplus_{0\leq i,j<m} \mathbb{C}[x,y]/\langle x - \alpha_{2i,2j+1}, y - \beta_{2i,2j+1}\rangle \\
& \oplus \bigoplus_{0\leq i,j<m} \mathbb{C}[x,y]/\langle x - \alpha_{2i+1,2j}, y - \beta_{2i+1,2j}\rangle \\
& \oplus \bigoplus_{0\leq i,j<m} \mathbb{C}[x,y]/\langle x - \alpha_{2i+1,2j+1}, y - \beta_{2i+1,2j+1}\rangle \\
\to \quad & \bigoplus_{0\leq i,j<m} \mathbb{C}[x,y]/\langle x - \alpha_{i,j}, y - \beta_{i,j}\rangle
\end{aligned}$$

The resulting algorithm has the form

$$\begin{aligned} P(&\mathrm{DTT}_{m\times m}(\tfrac{2}{3}, \tfrac{2}{3}) \oplus \mathrm{DTT}_{m\times m}(0,0) \oplus \mathrm{DTT}_{m\times m}(\tfrac{2}{3}\omega_3, \tfrac{2}{3}\omega_3^2) \\ &\oplus \mathrm{DTT}_{m\times m}(\tfrac{2}{3}\omega_3^2, \tfrac{2}{3}\omega_3))(\mathrm{DTT}_{2\times 2} \otimes \mathrm{I}_{m^2})B \end{aligned} \tag{23}$$

where B is an initial base change which can be performed using $O(n)$ operations and P is a permutation. The exact form will be derived elsewhere.

6. REFERENCES

[1] M. Püschel and J. M. F. Moura, "The Algebraic Approach to the Discrete Cosine and Sine Transforms and their Fast Algorithms," *SIAM Journal of Computing*, vol. 32, no. 5, pp. 1280–1316, 2003.

[2] M. Püschel and J. M. F. Moura, "The Discrete Trigonometric Transforms and Their Fast Algorithms: An Algebraic Symmetry Perspective," in *Proc. 10th IEEE DSP Workshop*, 2002, pp. 268–273.

[3] M. Püschel, "Cooley-Tukey FFT like Algorithms for the DCT," in *Proc. ICASSP*, 2003, pp. 501–504.

[4] T. S. Chihara, *An Introduction to Orthogonal Polynomials*, Gordon and Breach, 1978.

[5] T. Koornwinder, "Orthogonal polynomials in two variables which are eigenfunctions of two algebraically independent partial differential operators (part III)," *Indag. Math.*, vol. 36, pp. 357–369, 1974.

[6] P. E. Ricci, "An iterative property of chebyshev polynomials of the first kind in several variables," *Rend. Math. Appl.*, vol. 6, no. 4, pp. 555–563, 1986.