# HAAR FILTER BANKS FOR 1-D SPACE SIGNALS

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### ABSTRACT

We derive the Haar filter bank for 1-D space signals, based on our recently introduced framework for 1-D space signal processing, termed this way since it is built on a symmetric space shift operation in contrast to the directed time shift operation. The framework includes the proper notions of signal and filter spaces, "$z$-transform," convolution, and Fourier transform, each of which is different from their time equivalents. In this paper, we extend this framework by deriving the proper notions of a Haar filter bank for space signal processing, and show that it has a similar yet different form compared to the time case. Our derivation also sheds light on the nature of filter banks and makes a case for viewing them as projections on subspaces rather than as based on filters.

***Index Terms***— Wavelet transforms, Haar transforms, spectral analysis, Fourier transforms, algebra

## 1. INTRODUCTION

In the design and analysis of filter banks, 1-D infinite discrete signals are usually assumed to be *time signals*. That is, they are implicitly placed on an infinite line of equidistant time points. Furthermore, this line is directed, since there is an inherent understanding of direction in time, from "past" to "future": Fig. 1(a) visualizes this directed time model. It gives a natural meaning to crucial signal processing concepts including time delay and advance, linear convolution, and Fourier transform.

Many tools have been developed to analyze time signals. For example, the $z$-transform allows us to work with signals as Laurent series in $x = z^{-1}$. The spectrum of a signal is given by the discrete-time Fourier transform (DTFT), which in this case amounts to the evaluation of the $z$-transform of a signal on the unit circle $e^{j\omega}$, $\omega \in (-\pi, \pi]$. Finally, the decomposition of a signal into components that correspond to different levels of detail (by considering only specific frequencies present in the signal) is performed by appropriate filter banks. These concepts are summarized in Table 1.


(a) 1-D time model (b) 1-D space model ($C = V$)

**Fig. 1**. Visualization of signal models.

**1-D space signal processing.** A similar, albeit less well-known, approach to signal processing places signals on an undirected line of points, infinite only at one side of the origin [1, 2, 3]. There is no concept of direction, provided proper boundary conditions are specified. Such signals are called *1-D infinite space signals*, where "space" is used to emphasize the lack of inherent direction. Fig. 1(b) visualizes one of the signal models for undirected space signals.

The associated notion of the "$z$-transform" is now the $C$-transform, where $C \in \{T, U, V, W\}$ is one of four possible sequences of Chebyshev polynomials. The corresponding Fourier transform, called the discrete-space Fourier transform, evaluates the $C$-transform at $\cos(\omega)$, $\omega \in [0, \pi]$.

**Contribution of this paper.** One concept is missing, however: the definition and structure of a filter bank for space signals. The purpose of this paper is to expand the theory of 1-D infinite space signals with a proper definition of the decomposition of signals into frequency components, as well as the design of appropriate filter banks. As concrete example, we develop the proper notions of Haar filter banks for infinite space signals. We explain how the decomposition is performed through projections onto signal subspaces, and demonstrate that it can be implemented with properly designed time-varying filter banks.

**Organization.** Section 2 introduces 1-D space signal processing and compares it with the well-known 1-D time signal processing. Section 3 shows one possible derivation of the standard Haar filter bank for time signals. The same procedure is then used to derive the equivalent filter banks for 1-D space signals in Section 4. Section 5 summarizes the results presented in this paper.

## 2. BACKGROUND

We provide the algebraic background on signal processing as developed in the algebraic signal processing theory (ASP) [1] — a generalization of the linear signal processing (SP) as well as an axiomatic approach to SP based on the concept of a signal model defined below. Different signal models correspond to different notions of signal and filter spaces, $z$-transform, shift, Fourier transform, and other SP concepts. We focus our discussion on 1-D time signals and the nonstandard 1-D space signals.

We then briefly discuss an algebraic interpretation of the concept of the signal decomposition into components, and the implementation of this decomposition with filter banks by projecting a signal onto subspaces of signals that represent different frequency bands. We illustrate this concept with the concrete derivation of Haar filter banks for space signals.

**Algebra (filter space).** A vector space $\mathcal{A}$ that also allows for multiplication of its elements with each other, and supports the distributive law, is called an *algebra*. Examples include the sets of complex numbers $\mathbb{C}$ and complex polynomials in one variable $\mathbb{C}[x]$. In SP, the filter space is usually assumed to be an algebra (examples are below). Hence, we denote elements of $\mathcal{A}$ with $h$.

**Module (signal space).** Given an algebra $\mathcal{A}$, a (left) $\mathcal{A}$-module is a vector space $\mathcal{M}$ that admits a (left) multiplication of elements $s \in \mathcal{M}$ by elements $h \in \mathcal{A}$—$hs \in \mathcal{M}$—such that the distributive law holds. In SP, the signal space is usually assumed to be an $\mathcal{A}$-module, where $\mathcal{A}$ is the associated filter space, and the operation of $\mathcal{A}$ on $\mathcal{M}$ is filtering. We use $s$ to denote elements of $\mathcal{M}$.

**Signal model.** Signals do not arise as elements of modules, but (in the discrete case) as sequences of numbers $\mathbf{s} = (s_n)_{n \in I} \in V$

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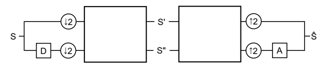
| | 1-D infinite time | 1-D infinite space |
|---|---|---|
| "$z$-transform" | $\mathbf{s} \mapsto s = \sum_{k\in\mathbb{Z}} s_k x^k$ | $\mathbf{s} \mapsto s = \sum_{k\in\mathbb{N}} s_k C_k$ |
| Fourier transform | $\omega \mapsto s(e^{iw})$ | $\omega \mapsto s(\cos w)$ |
| Filter bank |  | **?** |

**Table 1**. Basic concepts of 1-D infinite time and 1-D infinite space signal processing theory.

over some index domain, where $V$ is a vector space. The purpose of the *signal model* is to assign a notion of filtering to $V$. Formally, a *signal model* for a vector space $V$ is a triple $(\mathcal{A}, \mathcal{M}, \Phi)$, where $\mathcal{A}$ is a chosen filter algebra, $\mathcal{M}$ an associated signal $\mathcal{A}$-module, and $\Phi$ is a bijective mapping from $V$ to $\mathcal{M}$. $\Phi$ generalizes the concept of a $z$-transform as we will see below.

ASP is axiomatically built on the concept of the signal model. Once a signal model is defined, other concepts, such as convolution, spectrum, and Fourier transform, are automatically defined but take different forms for different models.

We illustrate this abstract discussion with examples: the infinite discrete-time SP, and the nonstandard infinite discrete-space SP. The goal of this paper is to derive the equivalent of Haar filter banks for the latter.

**1-D time signal model.** The signal model commonly adopted for infinite discrete time SP is for finite-energy sequences $V = \ell^2(\mathbb{Z})$. It is given by (we set $x = z^{-1}$)

$$\begin{aligned} \mathcal{A} &= \{\textstyle\sum_{n\in\mathbb{Z}} h_n x^n \mid \mathbf{h} = (\dots, h_{-1}, h_0, h_1, \dots) \in \ell^1(\mathbb{Z})\}, \\ \mathcal{M} &= \{\textstyle\sum_{n\in\mathbb{Z}} s_n x^n \mid \mathbf{s} = (\dots, s_{-1}, s_0, s_1, \dots) \in \ell^2(\mathbb{Z})\}, \\ \Phi &: \ \ell^2(\mathbb{Z}) \to \mathcal{M},\ \mathbf{s} \mapsto s = \textstyle\sum_{n\in\mathbb{Z}} s_n x^n. \end{aligned} \tag{1}$$

$\Phi$ is the standard $z$-transform. This signal model is a *time* model because of the directed operation of the shift operator $x \in \mathcal{A}$ on the basis elements $x^n$ of $\mathcal{M}$: $x \cdot x^n = x^{n+1}$. This operation is captured in the visualization of the model in Fig. 1(a). In fact, $p_n = x^n$ is the unique solution of the recurrence

$$p_{n+1} = x p_n, \tag{2}$$

with $p_0 = 1$. The basis in $\mathcal{A}$ consists of $k$-fold time shifts $\{x^k\}_{k\in\mathbb{Z}}$.

The associated Fourier transform is the discrete-time Fourier transform (DTFT) which maps elements $s = s(x) \in \mathcal{M}$ to functions on the unit circle $e^{j\omega}$, $\omega \in (-\pi, \pi]$:

$$\mathcal{F} :\ s = s(x) \mapsto s(e^{i\omega}) = \sum_{n\in\mathbb{Z}} s_n e^{i\omega n}.$$

Accordingly, the frequency response of a filter $h = \sum_{k\in\mathbb{Z}} h_k x^k$ is given by $h(e^{i\omega}) = \sum_{k\in\mathbb{Z}} h_k e^{i\omega k}$.

**1-D space signal model.** In [2, 1] we defined infinite discrete space models, which are derived from a different notion of shift operation, namely a symmetric shift $x \cdot p_n(x) = \frac{1}{2}(p_{n-1}(x) + p_{n+1}(x))$, which yields the recurrence

$$p_{n+1} = 2x p_n(x) - p_{n-1}(x), \tag{3}$$

with $p_0 = 1$ for normalization. The solution to this recurrence is exactly the Chebyshev polynomials,[1] $p_n = C_n$ and there are choices depending on the choice of $p_1 = C_1$. We consider the four cases $C \in \{T, U, V, W\}$ overviewed in Table 2. Note that in each case the sequence of polynomials has a symmetry; hence the resulting signal model will be only for right-sided signals.

The $k$-fold space shift is in each case given by $T_k(x)$, since $T_k C_n = \frac{1}{2}(C_{n-k} + C_{n+k})$.

As a result we obtain the following four signal models for $V = \ell^2(\mathbb{N})$, $C \in \{T, U, V, W\}$:

$$\begin{aligned} &\mathcal{A} = \{h = \textstyle\sum_{k\geq 0} h_k T_k(x) \mid \mathbf{h} \in \ell^1(\mathbb{N})\}, \\ &\mathcal{M} = \{s = \textstyle\sum_{n\geq 0} s_n C_n(x) \mid \mathbf{s} \in \ell^2(\mathbb{N})\}, \\ &\Phi :\ \ell^2(\mathbb{N}) \to \mathcal{M},\ \mathbf{s} \mapsto \textstyle\sum_{n\geq 0} s_n C_n(x). \end{aligned} \tag{4}$$

We call $\Phi$ the $C$-transform but will replace $C$ by either $T, U, V$, or $W$, when appropriate, and accordingly refer to the $T$-, $U$-, $V$-, or $W$-transform.

The symmetric shift yields the visualization in Fig. 1(b). For $C = V$, we have $V_{-1} = V_0$, which explains the looping edge at the left boundary.

The associated Fourier transform is the discrete-space Fourier transform that maps elements $s = s(x) \in \mathcal{M}$ to functions on the interval $[-1, 1]$, parameterized by $\cos\omega$, $\omega \in [0, \pi]$.

$$\mathcal{F} :\ s = s(x) \mapsto s(\cos\omega) = \sum_{n\in\mathbb{N}} s_n C_n(\cos\omega).$$

The frequency response of a filter $h = \sum_{k\geq 0} h_k T_k(x)$ is given by $h(\cos\omega) = \sum_{k\in\mathbb{N}} h_k T_k(\cos\omega)$. Both can be evaluated easily using the closed form of $C_n$ shown in Table 2.

**Signal decomposition and projections.** Filter banks are used to decompose a signal into components of different level of detail. Each such component contains only a certain band of frequencies present in the input signal, and it is common to view such a decomposition as being performed with bandpass filtering [4, 5]. For example, a usual 2-channel filter bank uses low-pass and high-pass filters in combination with up- and downsamplers to produce the "coarse" and "detailed" components.

To achieve our goal of deriving filter banks for space signals, we have to choose a different interpretation.

Namely, we view filter banks as performing projections of signals onto subspaces of low-frequency and high-frequency signals [6]. In the time case, the scalar product that is used for computation of the projections can be expressed through convolution, and is hence implemented directly with filtering. While it seems intuitive, this is not the case for other signal models, such as the space signal model, since they have different associated notions of convolution, while the notions of scalar product and projection remain the same. In Section 4 this will become clear, when we construct the Haar filter banks for space signals by computing projections of signals onto properly designed subspaces.

[1] Chebyshev polynomials $C_k$ are the polynomials that satisfy the two-term recurrence $C_{k+1} = 2xC_k - C_{k-1}$. Hence, the whole sequence of polynomials is determined by $C_0$ and $C_1$. By setting $x = \cos\theta$, Chebyshev polynomials can also be expressed in their trigonometric closed form as functions of $\theta$. These and other properties are shown in Table 2.

|  | $C_0,\ C_1$ | Closed form for $C_n$ | Symmetry | $C_n(1)$ | $C_n(-1)$ |
|---|---|---|---|---|---|
| $T$ | $1, x$ | $\cos{(n\theta)}$ | $T_{-n} = T_n$ | $1$ | $(-1)^n$ |
| $U$ | $1, 2x$ | $\frac{\sin{(n+1)\theta}}{\sin\theta}$ | $U_{-n} = -U_{n-2}$ | $n+1$ | $(-1)^n(n+1)$ |
| $V$ | $1, 2x-1$ | $\frac{\cos{(n+\frac{1}{2})\theta}}{\cos\frac{\theta}{2}}$ | $V_{-n} = V_{n-1}$ | $1$ | $(-1)^n(2n+1)$ |
| $W$ | $1, 2x+1$ | $\frac{\sin{(n+\frac{1}{2})\theta}}{\sin\frac{\theta}{2}}$ | $W_{-n} = -W_{n-1}$ | $2n+1$ | $(-1)^n$ |

**Table 2**. Chebyshev polynomials, symmetry, and values used for the derivation of space Haar filter banks.

## 3. 1-D TIME HAAR FILTER BANKS

We derive the standard Haar filter banks for the time signal model (1) by identifying suitable subspaces and associated projections. We use the same steps in the next section to derive Haar filter banks for the space signal model (4).

We consider the signal model (1), where, for a signal $s = \sum_k s_k x^k \in \mathcal{M}$, we want to compute its "coarse" approximation $s'$ and "detailed" complement $s''$ . We do so by projecting the signal $s$ onto subspaces $\mathcal{M}_l$ and $\mathcal{M}_h$ that consist of low-frequency and high-frequency signals, respectively. We show that $\mathcal{M}_l \oplus \mathcal{M}_h = \mathcal{M}$, hence such a decomposition is a complete representation of signals in $\mathcal{M}$.

**Subspace construction.** We impose the following structure on the subspaces: the basis of $\mathcal{M}_l$ is $\phi = \{\phi_n\}_{n\in\mathbb{Z}} = \{a_n x^{2n} + b_n x^{2n+1}\}_{n\in\mathbb{Z}}$; and the basis of $\mathcal{M}_h$ is $\psi = \{\psi_n\}_{n\in\mathbb{Z}} = \{c_n x^{2n} + d_n x^{2n+1}\}_{n\in\mathbb{Z}}$. To satisfy the low- and high-frequency conditions, we require the spectrum of basis functions $\phi_n$ and $\psi_n$ to disappear, correspondingly, at the highest ($\omega = \pi$) and lowest ($\omega = 0$) frequencies:

$$\begin{cases}\phi_n(e^{\pi i}) = a_n e^{2n\pi i} + b_n e^{(2n+1)\pi i} = 0\\ \psi_n(e^{0i}) = c_n e^{2n0i} + d_n e^{(2n+1)0i} = 0\end{cases}. \tag{5}$$

The resulting conditions on the coefficients are $a_n = b_n$ and $c_n = -d_n$. Hence, the required bases are

$$\begin{aligned}\phi &= \{a_n x^{2n} + a_n x^{2n+1}\}_{n\in\mathbb{Z}}, &(6)\\ \psi &= \{c_n x^{2n} - c_n x^{2n+1}\}_{n\in\mathbb{Z}}. &(7)\end{aligned}$$

It immediately follows that the above bases span the whole signal space $\mathcal{M} = \langle\phi, \psi\rangle$ and that the respective subspaces have trivial intersection $\mathcal{M}_l \cap \mathcal{M}_h = \{0\}$. Thus, $\mathcal{M} = \mathcal{M}_l \oplus \mathcal{M}_h$.

**Computation of the projections.** To compute the projections of the signal $s$ onto $\mathcal{M}_l$ and $\mathcal{M}_h$, we:

1. Find dual bases $\tilde{\phi} = \{\tilde{\phi}_n\}$ and $\tilde{\psi} = \{\tilde{\psi}_n\}$;
2. Compute scalar products $\langle s, \tilde{\phi}_n\rangle$ and $\langle s, \tilde{\psi}_n\rangle$;
3. Construct projections $s' = \sum_n \langle s, \tilde{\phi}_n\rangle\phi_n \in \mathcal{M}_l$ and $s'' = \sum_n \langle s, \tilde{\psi}_n\rangle\psi_n \in \mathcal{M}_h$.

**Dual bases.** In addition to the usual biorthogonality requirements on $\phi$ and $\tilde{\phi}$, $\psi$ and $\tilde{\psi}$, we require $\phi \cup \psi$ and $\tilde{\phi} \cup \tilde{\psi}$ to be biorthogonal, since $\phi \cup \psi$ is a basis of $\mathcal{M}$. Altogether, the bases and their dual counterparts must satisfy the following conditions:

$$\begin{cases}\langle\phi_k, \tilde{\phi}_m\rangle = \langle\psi_k, \tilde{\psi}_m\rangle = \delta_{k-m}\\ \langle\phi_k, \tilde{\psi}_m\rangle = \langle\tilde{\phi}_k, \psi_m\rangle = 0\end{cases}. \tag{8}$$

Equations (6)-(7) and conditions (8) yield the dual bases

$$\tilde{\phi}_n = \frac{1}{2a_n}x^{2n} + \frac{1}{2a_n}x^{2n+1},\ \tilde{\psi}_n = \frac{1}{2c_n}x^{2n} - \frac{1}{2c_n}x^{2n+1}.$$

**Projections.** It follows that the projections of $s$ on subspaces $\mathcal{M}_l$ and $\mathcal{M}_h$ are

$$\begin{aligned}s' &= \sum_n (\frac{1}{2a_n}s_{2n} + \frac{1}{2a_n}s_{2n+1})\phi_n, &(9)\\ s'' &= \sum_n (\frac{1}{2c_n}s_{2n} - \frac{1}{2c_n}s_{2n+1})\psi_n. &(10)\end{aligned}$$

**Implementation.** From (9)-(10), it follows that the projections can be implemented with the well-known Haar filter bank (Fig. 3 shows the so-called polyphase version of the Haar filter bank).

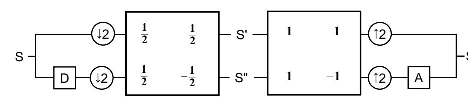



**Fig. 2**. Haar filter bank for 1-D time signals (with $a_n = c_n = 1$).

## 4. 1-D SPACE HAAR FILTER BANK

We now follow the same procedure as for the time model to construct the Haar filter bank for the space signal models in (4). For the detailed derivation, we focus on the case $C = V$ and signal space $\mathcal{M} = \{\sum_k s_k V_k\}$, where $V_k$ are the Chebyshev polynomials of the third kind. Filter banks for signal spaces that correspond to other Chebyshev polynomials are constructed analogously; the complete list of bases and example filter banks are provided in Table 3.

As in Section 3, for any signal $s = \sum_k s_k V_k \in \mathcal{M}$, we compute its "coarse" approximation $s'$ and "detailed" complement $s''$ by projecting the signal $s$ onto the low-frequency and high-frequency signal subspaces $\mathcal{M}_l$ and $\mathcal{M}_h$. We also require $\mathcal{M}_l \oplus \mathcal{M}_h = \mathcal{M}$ to make the decomposition a complete representation of signals in $\mathcal{M}$.

**Subspace construction.** We impose the following structure on the subspaces: the basis of $\mathcal{M}_l$ is $\phi = \{\phi_n\}_{n\in\mathbb{N}} = \{a_n V_{2n} + b_n V_{2n+1}\}_{n\in\mathbb{N}}$ and the basis of $\mathcal{M}_h$ is $\psi = \{\psi_n\}_{n\in\mathbb{N}} = \{c_n V_{2n} + d_n V_{2n+1}\}_{n\in\mathbb{N}}$. The requirements that the spectra of basis functions $\phi_n$ and $\psi_n$ disappear at the highest ($\omega = \pi$) and lowest ($\omega = 0$) frequencies for space signals translate into equations

$$\begin{cases}a_n V_{2n}(\cos\pi) + b_n V_{2n+1}(\cos\pi) = 0\\ c_n V_{2n}(\cos 0) + d_n V_{2n+1}(\cos 0) = 0\end{cases}. \tag{11}$$

Using the corresponding values from Table 2, we compute the resulting conditions on the coefficients: $b_n = \frac{4n+1}{4n+3}a_n$ and $d_n = -c_n$. Hence, the bases for $\mathcal{M}_l$ and $\mathcal{M}_h$ are

$$\begin{aligned}\phi &= \{a_n V_{2n} + \frac{4n+1}{4n+3}a_n V_{2n+1}\}_{n\in\mathbb{N}}, &(12)\\ \psi &= \{c_n V_{2n} - c_n V_{2n+1}\}_{n\in\mathbb{N}}. &(13)\end{aligned}$$

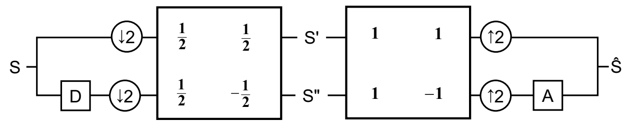
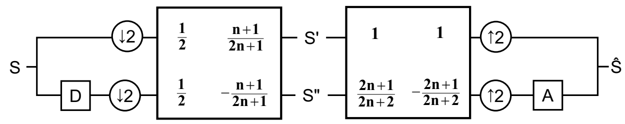
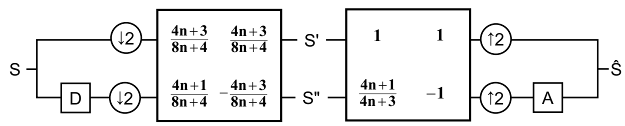
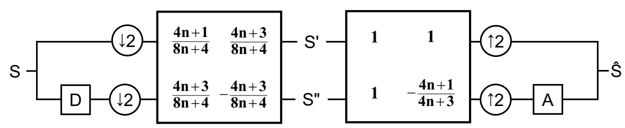
| $C$ | | Haar Basis | Haar Filter Bank |
|---|---|---|---|
| $T$ | $\phi_n$ | $a_n T_{2n} + a_n T_{2n+1}$ |  |
| | $\psi_n$ | $c_n T_{2n} - c_n T_{2n+1}$ | |
| | $\tilde{\phi}_n$ | $\frac{1}{2a_n} T_{2n} + \frac{1}{2a_n} T_{2n+1}$ | |
| | $\tilde{\psi}_n$ | $\frac{1}{2c_n} T_{2n} - \frac{1}{2c_n} T_{2n+1}$ | |
| $U$ | $\phi_n$ | $a_n U_{2n} + \frac{2n+1}{2n+2} a_n U_{2n+1}$ |  |
| | $\psi_n$ | $c_n U_{2n} - \frac{2n+1}{2n+2} c_n U_{2n+1}$ | |
| | $\tilde{\phi}_n$ | $\frac{1}{2a_n} U_{2n} + \frac{n+1}{(2n+1)a_n} U_{2n+1}$ | |
| | $\tilde{\psi}_n$ | $\frac{1}{2c_n} U_{2n} - \frac{n+1}{(2n+1)c_n} U_{2n+1}$ | |
| $V$ | $\phi_n$ | $a_n V_{2n} + \frac{4n+1}{4n+3} a_n V_{2n+1}$ |  |
| | $\psi_n$ | $c_n V_{2n} - c_n V_{2n+1}$ | |
| | $\tilde{\phi}_n$ | $\frac{4n+3}{(8n+4)a_n} V_{2n} + \frac{4n+3}{(8n+4)a_n} V_{2n+1}$ | |
| | $\tilde{\psi}_n$ | $\frac{4n+1}{(8n+4)c_n} V_{2n} - \frac{4n+3}{(8n+4)c_n} V_{2n+1}$ | |
| $W$ | $\phi_n$ | $a_n W_{2n} + a_n W_{2n+1}$ |  |
| | $\psi_n$ | $c_n W_{2n} - \frac{4n+1}{4n+3} c_n W_{2n+1}$ | |
| | $\tilde{\phi}_n$ | $\frac{4n+1}{(8n+4)a_n} W_{2n} + \frac{4n+3}{(8n+4)a_n} W_{2n+1}$ | |
| | $\tilde{\psi}_n$ | $\frac{4n+3}{(8n+4)c_n} W_{2n} - \frac{4n+3}{(8n+4)c_n} W_{2n+1}$ | |

**Table 3**. Bases for subspaces $\mathcal{M}_l$ and $\mathcal{M}_h$ and associated Haar filter banks (with $a_n = 1$ and $c_n = 1$) for the four space signal models in (4).

Note that the basis functions $\phi_k$ and $\psi_n$ are independent of each other. Moreover, the original basis $\{V_n\}_{n\in\mathbb{N}}$ can be expressed in terms of $\phi_n$ and $\psi_n$: assuming $a_n = 1$ and $c_n = 1$, $V_{2n} = \frac{4n+3}{8n+4}\phi_n + \frac{4n+1}{8n+4}\psi_n$ and $V_{2n+1} = \frac{4n+3}{8n+4}\phi_n - \frac{4n+3}{8n+4}\psi_n$. Thus, $\phi \cup \psi$ is a basis for the signal space $\mathcal{M}$. Since $\mathcal{M}_l \cap \mathcal{M}_h = \{0\}$, it immediately follows that $\mathcal{M} = \mathcal{M}_l \oplus \mathcal{M}_h$.

**Computation of the projections.** To compute the projections of the signal $s$ onto $\mathcal{M}_l$ and $\mathcal{M}_h$, we follow the same procedure as in Section 3: construct dual bases $\tilde{\phi}$ and $\tilde{\psi}$, and then compute scalar products $\langle s, \tilde{\phi}_n\rangle$ and $\langle s, \tilde{\psi}_n\rangle$ to find the projection coefficients.

**Dual bases.** From eqs.(12)-(13) and (8) we derive the dual bases

$$\begin{aligned}\tilde{\phi}_n &= \frac{4n+3}{(8n+4)a_n} V_{2n} + \frac{4n+3}{(8n+4)a_n} V_{2n+1},\\ \tilde{\psi}_n &= \frac{4n+1}{(8n+4)c_n} V_{2n} - \frac{4n+3}{(8n+4)c_n} V_{2n+1}.\end{aligned}$$

**Projections.** It follows that the projections of the signal $s$ onto subspaces $\mathcal{M}_l$ and $\mathcal{M}_h$ are

$$s' = \sum_n \left(\frac{4n+3}{(8n+4)a_n} s_{2n} + \frac{4n+3}{(8n+4)a_n} s_{2n+1}\right)\phi_n, \quad (14)$$

$$s'' = \sum_n \left(\frac{4n+1}{(8n+4)c_n} s_{2n} - \frac{4n+3}{(8n+4)c_n} s_{2n+1}\right)\psi_n. \quad (15)$$

**Implementation.** The projections (14)-(15) can be implemented with the time-varying filter bank shown in Table 3.

**Filter banks for other 1-D space signal models.** Derivation of the subspaces $\mathcal{M}_l$ and $\mathcal{M}_h$ and computation of projections for other signal spaces $\mathcal{M} = \{\sum_k s_k C_k\}$, where $C_k$ denote Chebyshev polynomial of any of the four kinds, is analogous to the derivation we just saw. Bases for $\mathcal{M}_l$ and $\mathcal{M}_h$, their corresponding dual bases, and example filter banks (for parameter values $a_n = 1$ and $c_n = 1$) are included in Table 3. Observe that as $n \to \infty$, the index-dependent filter coefficients converge to the index-independent coefficients of the time Haar filter bank.

## 5. CONCLUSIONS

We derived Haar filter banks for 1-D space signal models. This result shows that meaningful SP frameworks can be built on notions of filtering and Fourier transform different from the standard time SP. In doing so, it also provides a deeper understanding of the nature of filter banks; namely, to make our derivation possible we needed to view filter banks as subspace projections rather than as based on filters. Finally, this paper is a first step in expanding the algebraic signal processing theory to include filter banks.

## 6. REFERENCES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